Quantum Operators & Gates

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Concept	Properties / Definitions	Examples / Representations
Hermitian Operator	$H^\dagger = H$. It has real eigenvalues. Examples include observables like energy.	σ_x, σ_z, $H = \frac{p^2}{2m} + V(x)$
Anti-Hermitian	$A^\dagger = -A$. It has imaginary eigenvalues. e^A is unitary.	$A = iH$ where H is Hermitian
Unitary Operator	$U^\dagger U = I$. It preserves norm. Reversible.	e^{-iHt}, CNOT, rotation gates
Euler Decomposition	Any 1-qubit unitary can we written as: $U = R_z(\phi)R_y(\theta)R_z(\lambda)$	Used in Qiskit/IBM Q
Parameters	$U(N)$: N^2 real parameters $SU(N)$: $N^2{-}1$ real parameters Hermitian: N^2 real parameters	$SU(2)$: 3 real parameters Hermitian 2×2: 4 real parameters
Orthogonal Matrix	$O^T O = I$. Real unitaries. Rotations/reflections.	$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
Generators	$U = e^{iH}$ where H is Hermitian *generator*	$H = \sigma_x \Rightarrow U = e^{-iHt}$
Pauli Matrices	Basis of SU(2). Hermitian + unitary.	$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Clifford Group	Maps Paulis to Paulis. Efficient for error correction.	H, S, CNOT; not universal alone
Gate Decomposition	Any unitary $\approx$ finite universal gate set	H, T, CNOT, Solovay-Kitaev theorem
Universality	Gate set can approximate any $U \in SU(2^n)$	Clifford + T, or {H, T, CNOT}
SU(2^n) Coverage	Cliffords $\subset$ dense subset of $SU(2^n)$. Need non-Cliffords.	T gate, Toffoli, arbitrary rotations

Practice Questions

1. Evaluate the expectation value of $\sigma_x, \sigma_y, \sigma_z$ in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.
2. Is $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ unitary?
3. Diagonalize σ_x, and find its eigenvalues.
4. Count independent real parameters in SU(3).
5. Is $e^{i\sigma_y}$ unitary? Is it hermitian?
6. What are the allowed eigenvalues of operators which are both hermitian and unitary. Give some examples.
7. Write the matrix representation of H gate in $\{|+\rangle, |-\rangle\}$ basis and verify its unitarity.
8. Find Euler angles for $R_y(\theta)$.
9. Simplify $e^{-i\pi\sigma_x/2}$.
10. Is the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ Hermitian?
11. Give example of a non-unitary matrix?
12. Prove that $A = iH$ is anti-Hermitian if H is hermitian.
13. Can a real matrix be unitary?
14. Give example of a 2×2 orthogonal matrix.
15. Prove $U^\dagger U = I$ for $U = e^{-iHt}$
16. What is global phase in $e^{i\theta} I$?
17. Prove $R_z(\phi)$ is unitary
18. Count independent real parameters in 4×4 Hermitian matrix.
19. Which gate completes Clifford for universality?
20. Why is T gate needed?
21. Simplify $e^{-i\pi\sigma_x/4}$.
22. Diagonalize $\sigma_x + \sigma_y$
23. Why does SU(2) exclude global phase?
24. Show that an arbitrary single-qubit unitary can be written in the form
$$\mathcal{U} = e^{i\alpha} R_{\hat{n}}(\theta),$$
where, $R_{\hat{n}}(\theta) = e^{-i(\theta/2)\hat{n}.\vec{\sigma}} = \cos(\theta/2)\mathbb{I} - i\sin(\theta/2)(\hat{n}_x\sigma_x + \hat{n}_y\sigma_y + \hat{n}_z\sigma_z)$.
25. Show that any arbitrary single qubit unitary can be expressed using rotations in the z and y axes and a phase shift in the form
$$\mathcal{U} = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta).$$

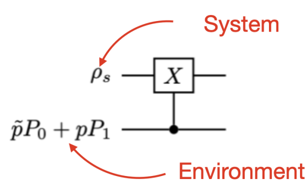
Kraus Operators, Partial Trace, and Entanglement

Topic	Definition and Explanation	Exercises and Examples
Mixed States	Quantum states are described by density matrices ρ. A state is **mixed** if it cannot be written as a pure state $\|\psi\rangle\langle\psi\|$.	*Ex:* $\rho = \frac{1}{2}\|0\rangle\langle 0\| + \frac{1}{2}\|1\rangle\langle 1\|$
Purity	A measure of how mixed a state is: $\mathrm{Tr}(\rho^2)$. • $= 1$: pure, < 1: mixed • Maximally mixed: $\rho = \frac{\mathbb{I}}{d} \Rightarrow \mathrm{Tr}(\rho^2) = \frac{1}{d}$	*Q:* Show $\mathrm{Tr}(\rho^2) = 1$ for pure states. *Q:* Compute for $\rho = \frac{\mathbb{I}}{2}$.
Entanglement (Pure)	A pure state is entangled if it cannot be written as $\|\psi\rangle_{AB} = \|\phi\rangle_A \otimes \|\chi\rangle_B$. Entangled pure states yield mixed reduced states.	*Q:* Is $\|\psi\rangle = \frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$ entangled? Compute ρ_A.
Entanglement (Mixed)	Mixed states are entangled if not expressible as: $\rho_{AB} = \sum_j p_j \rho_j^A \otimes \rho_j^B$.	*Q:* Example of entangled mixed state? *Q:* If $\rho_A = \rho_B = \frac{\mathbb{I}}{2}$, is it separable?
Schmidt Decomposition (SD)	The SD of any pure bipartite state is given as: $\|\psi\rangle = \sum_i \lambda_i \|u_i\rangle_A \otimes \|v_i\rangle_B$, if $\{\|u_i\rangle_A\}, \{\|v_i\rangle_B\}$ form a locally orthonormal basis. • $\lambda_i \geq 0$, $\sum \lambda_i^2 = 1$ • Schmidt rank = #nonzero λ_i • Rank 1 $\Leftrightarrow$ separable	*Q:* Schmidt decompose $\|\psi\rangle = \frac{1}{\sqrt{3}}(\|00\rangle + \|11\rangle + \|22\rangle)$. Is it already in Schmidt decomposed form? Q. Give a Schmidt decomposition of the state $\frac{1}{2}(\|00\rangle + \|01\rangle + \|10\rangle - \|11\rangle)$. What is its Schmidt rank? Q. Give a Schmidt decomposition of the state $\frac{1}{2}(\|00\rangle + \|01\rangle + \|10\rangle + \|11\rangle)$. What is its Schmidt rank?
Partial Trace	Traces out subsystem. From ρ_{AB}, define: $\rho_A = \mathrm{Tr}_B(\rho_{AB})$. Models information loss.	*Q:* For $\|\psi\rangle = \frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$, find ρ_A.
Kraus Operators	CPTP maps describe quantum operations: $\rho' = \sum_k K_k \rho K_k^\dagger$, $\sum K_k^\dagger K_k = \mathbb{I}$. Models environment-induced evolution.	*Q:* Show $K_0 = \sqrt{p}\,\mathbb{I}$, $K_1 = \sqrt{1-p}\,Z$ defines a CPTP map.
Stinespring Dilation	Any channel = unitary on larger system + partial trace: $\rho' = \mathrm{Tr}_E\left[U(\rho \otimes \|0\rangle_E\langle 0\|)U^\dagger\right]$	*Q:* Write Kraus ops for amplitude damping channel.

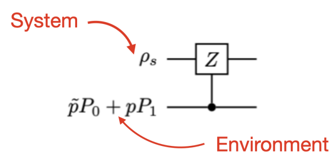
Practice Questions

1. True/False: Every state can be expressed in the form $|\psi\rangle\langle\psi|$.
2. What is the purity of $|+\rangle\langle +|$, $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$?
3. Give examples of 2-qubit pure product states.
4. Is $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ pure or mixed?
5. Show the Kraus operators $K_0 = \sqrt{p}\mathbb{I}$, $K_1 = \sqrt{1-p}Z$ correspond to a CPTP map.
6. Find the purity of $\rho = \frac{\mathbb{I}}{2}$.
7. For $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, compute ρ_A.
8. Find the Schmidt rank of $|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$.
9. Why is reduced state of entangled pure state mixed?
10. Prove $\mathrm{Tr}(\rho^2) = 1$ for pure ρ.
11. If the Schmidt rank of a state is 1, is it separable or entangled?
12. If the reduced density matrices of a bipartite state ρ_{AB} are $\rho_A = \rho_B = \frac{\mathbb{I}}{2}$, is the bipartite state ρ_{AB} separable or entangled?
13. What does partial trace mean physically?
14. Stinespring dilation for: $\rho \mapsto (1-p)\rho + pZ\rho Z$
15. Prove: Entropy of pure bipartite separable state = entropy of reduced state. Does it hold for bipartite entangled states too?
16. Is $\rho = \frac{1}{3}|\phi^+\rangle\langle\phi^+| + \frac{2}{3}\frac{\mathbb{I}_4}{4}$ entangled?
17. Show: for $|\psi\rangle = \sum_i \lambda_i |i\rangle_A |i\rangle_B$, Purity$(\rho_A) = \sum \lambda_i^4$.
18. Find the Kraus operators for the following CPTP maps (system and environment are highlighted):

    (a) Bit-flip channel ($\tilde{p} = 1 - p$):

    (b) Phase flip channel

    (c) Depolarising channel