Introduction to Quantum Measurement & Decoherence

Srijit Bhattacharjee
Department of Applied Sciences
IIIT Allahabad

1 Recap of Quantum Postulates

1. All possible information of a quantum system at a given instant t is encoded in a state vector $|\psi(t)\rangle$, it is an element of a Hilbert space $\mathcal{H}$ (a complete inner-product space).

2. Every measurable physical quantity (Observable) is is described by a Hermitian operator $\hat{\mathcal{A}}$, acting on $\mathcal{H}$.

3. Only possible outcome of measurement of $\hat{\mathcal{A}}$ is one of the eigenvalues of $\hat{\mathcal{A}}$.

4. When the observable $\hat{\mathcal{A}}$ is measured on a system in the normalised state $|\psi(t)\rangle$, the probability $\mathcal{P}(a_n)$ of obtaining an eigenvalue a_n is
$$\mathcal{P}(a_n) = |\langle u_n |\psi\rangle|^2$$
(for discrete spectrum and non-degenerate eigenvalues) The eigenkets $\{|u_n\rangle\}$ form a basis.

5. If the measurement of the physical quantity $\hat{\mathcal{A}}$ on the system in the state $|\psi(t)\rangle$ results the outcome a_n, the state of the system immediately after the measurement is:
$$\frac{P_n |\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$$
$P_n = |u_n\rangle\langle u_n|$, is the projector onto the eigensubspace associated to the eigenvalue a_n. The state does not change after the measurement.

6. Time evolution of the state $|\psi\rangle$ is governed by the Schrö dinger's equation
$$\frac{d}{dt}|\psi(t)\rangle = \frac{\hat{H}}{i\hbar}|\psi(t)\rangle$$

Example 1: A spin-2 system is an example of a 2-dimensional quantum system. The general state of such a system can be described by a state ket:

$$|\alpha\rangle = c_0|+\rangle + c_1|-\rangle,$$

with $|\pm\rangle$ are the spin-up and spin-down states of the system, and they are represented by culumn vectors $(1,0)^T$ and $(0,1)^T$ respectively. In general, the quantum state can be a linear combination of the two possible states. In quantum computation and quantum information it is known as a Qubit as they are the basic units of storing quantum information.

2 Density Operator, Pure states, Mixed states

Let us consider a single quantum state $|\psi\rangle$. To find the expectation value of the operator $\hat{\mathcal{A}}$, in this state, one first expresses the state as a linear combination of the eigenstates of $\hat{\mathcal{A}}$.

$$|\psi\rangle = \sum_n c_n |u_n\rangle$$

$$\langle \hat{\mathcal{A}} \rangle = \langle \psi | \hat{\mathcal{A}} | \, |\psi\rangle = \sum_n a_n |c_n|^2 = \sum_n a_n \mathcal{P}(a_n)$$

The interpretation of expectation value of an operator necessitates us to introduce the concept of 'ensemble'. The expectation value is the ensemble average of the results of the measurements.

A quantum system can be described as a single-ket vector when it is in a **pure state**. For example, a beam coming out of a Stern-Gerlach (SG) apparatus will have a definite polarization. Therefore, this pure state is always expressible as a coherent mixture of two states

$$|\alpha\rangle = c_0|+\rangle + c_1|-\rangle.$$

In this case, one can always find a definite relation between the coefficients c_0, c_1.

When a quantum system is not in a coherent combination, or when it is described by a statistical mixture of states, we call it is in a **mixed state**. In this case, the state is given by

$$|\psi\rangle = \sum_i w_i \, |\phi^{(i)}\rangle \, ,$$

with w_i denotes the probability that the system would be found in the substate $|\phi^{(i)}\rangle$. This is not to be confused with the representation of a state vector as a linear combination of eigenstates of an operator as $|\phi^{(i)}\rangle$s are not orthogonal to each other in general. Further, w_is are classical probabilities satisfying $\sum_i w_i = 1$. An example of this kind of incoherent mixture of states is a beam coming out of the oven of the SG apparatus, the beam here is a random mixture of possible polarizations.

Now, the expectation value of an operator $\hat{\mathcal{A}}$ (assuming that there are N subsystems available) is:

$$\langle \hat{\mathcal{A}} \rangle = \sum_{i=1}^{N} w_i \langle \phi^{(i)} | \hat{\mathcal{A}} | \phi^{(i)} \rangle.$$

To evaluate the expectation value, it is convenient to define an operator, namely the **density operator**:

$$\hat{\rho} = \sum_{i=1}^{N} w_i |\phi^{(i)}\rangle\langle\phi^{(i)}|$$

Considering a complete set of eigenkets $\{|b_k\rangle\}$ of operator $\hat{\mathcal{A}}$, and inserting completeness relations like $\sum_{k=1}^{d} |b_k\rangle \langle b_k|$, one gets ($d$ is the dimensionality of the state space corresponding to $|\phi^{(i)}\rangle = \sum_{k=1}^{d} c_k^{(i)} \, |b_k\rangle$):

$$\begin{aligned} \langle \hat{\mathcal{A}} \rangle &= \sum_{i=1}^{N} \sum_{k,l=1}^{d} w_i \, \langle b_k | \, \hat{\rho} | b_l \rangle \hat{\mathcal{A}}_{lk} \\ &= Tr(\hat{\rho}\hat{\mathcal{A}}) \end{aligned}$$

2.1 Properties of density operator:

$\hat{\rho}$ is Hermitian. For a pure state

$$\hat{\rho}^2 = \hat{\rho}.$$

Further, for a pure ensemble

$$Tr(\hat{\rho^2}) = 1.$$

For a mixed ensemble $\hat{\rho}^2 \neq \hat{\rho}$, and $Tr\hat{\rho}^2 \leq 1$. These relations can be used to distinguish a pure ensemble and a mixed ensemble.

Example 2: Density operator for the 2-level system with $c_0 = c_1 = \frac{1}{\sqrt{2}}$ is $|\alpha\rangle \langle\alpha|$.

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Check this is a pure state. Now consider an incoherent mixture of base states $|+\rangle$ and $|-\rangle$, with $w_1 = w_2 = \frac{1}{2}$. The density operator now becomes

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

One can check this is a mixed ensemble.

2.2 Population & Coherence:

The diagonal entries of the density matrix are known as *population*.

$$\hat{\rho}_{kk} = \sum_i w_i |c_k^{(i)}|^2.$$

It denotes the average probability of finding the system in the state $|b_k\rangle$. Population can only be zero if and only if all the $|c_k^{(i)}|^2$ are zero.

On the other hand, the off-diagonal entries are known as *coherences* as they signify the interference between different states of the system.

$$\hat{\rho}_{kl} = \sum_i w_i c_k^{(i)} c_l^{(i)*}.$$

If the system is in a coherent mixture of two states then $\hat{\rho}_{kl}$ will be non-zero indicating interference between the k^{th} and l^{th} states. Coherence terms are complex numbers and may add up to zero even if none of them are individually zero.

3 Decoherence and non-observability of interference pattern

There are issues related to quantum measurements and their interpretations that need further explanations. For example, why superposition is not observed in our classical world although every interaction fundamentally is quantum in nature? Why interferences are absent in classical measuring processes? How a quantum system collapses to a classical state after a measurement is done?

Decoherence phenomenon offers plausible answers to some of these questions. We recall the Young's double slit (DS) experiment with quantum particles. The interference pattern on the screen gets destroyed whenever one could obtain the information regarding the path or slit through which the particle passed through. There is an interesting trade-off between the interference pattern and the which-path information. Decoherence is a mechanism that captures this trade-off efficiently.

Within the Von Neumann 'ideal' measurement scheme one considers a system S and an environment E but the system is not disturbed by the environment. Initially let the system for DS experiment is in state $|s\rangle$ and the initial environment (before the measurement) is in state $|E_0\rangle$. The system-environment is in the state following state as the system has to be represented as linear a superposition of two possible states (particle passing through either slit 1 or 2):

$$|s\rangle \otimes |E_0\rangle = (c_1 |s_1\rangle + c_2 |s_2\rangle) \otimes |E_0\rangle$$

Due to the system-environment (SE) interaction, the state for the environment changes according to the following:

$$|s_1\rangle |E_0\rangle \to |s_1\rangle |E_1\rangle$$

$$|s_2\rangle |E_0\rangle \to |s_2\rangle |E_2\rangle$$

So, the SE composite state after the interaction becomes entangled,

$$|\psi\rangle = c_1 |s_1\rangle |E_1\rangle + c_2 |s_2\rangle |E_2\rangle \,.$$

The density operator for this joint system is $\hat{\rho}_{SE} |\psi\rangle \langle\psi|$. One can now trace-out the environment part to yield the reduced density matrix:

$$\hat{\rho}_S = Tr_E \ \hat{\rho}_{SE} = |c_1|^2 |s_1\rangle \langle s_1| + |c_2|^2 |s_2\rangle \langle s_2| + c_1 c_2^* |s_1\rangle \langle s_2| \langle E_2| E_1\rangle + c_1^* c_2 |s_2\rangle \langle s_1| \langle E_1| E_2\rangle. \tag{1}$$

If $\mathcal{P}(x)$ denotes the position probability density of finding the particle on a screen, then

$$\mathcal{P}(x) = \langle x|\hat{\rho}_S|x\rangle = |c_1|^2|\psi_1(x)|^2 + |c_2|^2|\psi_2(x)|^2 + 2Re\{c_1 c_2^* \ \psi_1(x)\psi_2(x)^* \langle E_2| E_1\rangle\} \tag{2}$$

The last term of the R.H.S. of Eq. (2) signifies the interference effect. If the paths are indistinguishable then $\langle E_2| E_1\rangle \neq 0$, and coherence prevails. If one gets which-path information then the paths are completely distinguishable and the states E_1, E_2 are orthogonal i.e. $\langle E_2| E_1\rangle = 0$. So the coherence is lost and the interference pattern becomes unobservable.

Many models of Decoherence exhibit an exponential decay of the correlation $\langle E_2| (t)|E_1(t)\rangle \propto e^{-t/\tau_d}$, due to large number of interactions between particles and environments. The decoherence time scale τ_d usually varies from millisecond to nano-second for different types of qubits (photonic, ion-traps, superconducting etc.). Decoherence is a barrier to efficiently design a quantum computer as the qubits tend to loose their 'quantumness' at an exponential rate.

Reference

1. Quantum Decoherence, by M. Schlosshauer; Physics Reports, Volume 831, 25 October 2019, Pages 1-57